

Linear-Quadratic Optimal Control Problems for Mean-Field Backward Stochastic Differential Equations with Jumps ^{*}

Maoning Tang^a and Qingxin Meng^{†a}

^aDepartment of Mathematical Sciences, Huzhou University, Zhejiang 313000, China

Abstract

This paper is concerned with a linear quadratic (LQ, for short) optimal control problem for mean-field backward stochastic differential equations (MF-BSDE, for short) driven by a Poisson random martingale measure and a Brownian motion. Firstly, by the classic convex variation principle, the existence and uniqueness of the optimal control is established. Secondly, the optimal control is characterized by the stochastic Hamilton system which turns out to be a linear fully coupled mean-field forward-backward stochastic differential equation with jumps by the duality method. Thirdly, in terms of a decoupling technique, the stochastic Hamilton system is decoupled by introducing two Riccati equations and a MF-BSDE with jumps. Then an explicit representation for the optimal control is obtained.

Keywords: Mean-Field; Optimal Control; Backward Stochastic Differential Equation; Adjoint Process

1 Introduction

Stochastic optimal control problems of mean-field type recently are extensively studied, due to their comprehensive practical applications especially in economics and finance. Different from the classic stochastic optimal control problem, the probability distribution of the state and the control are involved in the coefficients of the state equation and cost functional which leads to a time-inconsistent optimal control problem, so the dynamic programming principle (DPP) is not effective and many researchers try to solve this type of optimal control problems by discuss the stochastic maximum principle (SMP) instead of trying extensions of DPP. One can refer to [1-3],[6],[9], [12],[16], [17], [22],[23],[25-28] for more results on the stochastic maximum principles for different types of mean-field stochastic models.

In 2013, the continuous-time mean-field LQ problem for mean-field forward stochastic differential equation (MF-FSDE) were systematically studied by Yong [29], where the optimal control is represented as a state feedback form by introducing two Riccati differential equations. Since [29], many advances have been made on LQ control problem for stochastic system of mean-field type (cf., for example [8], [10], [13]-[15],[18-21],[26]). In 2016, Tang and Meng [24] extended continuous-time mean-field LQ control problem to jump diffusion system and established the corresponding theoretical results. It is well-knew that the adjoint equation of a controlled MF-FSDE is a MF-BSDE. So it is not until Buckdahn, Djehiche, Li and Peng [4]; Buckdahn, Li and Peng [5] established the theory of the mean-field BSDEs that the stochastic maximum principle for the optimal control system of mean-field type has become a popular topic. Since a BSDE is a well-defied dynamic system itself and has important applications in mathematical finance, it is necessary and natural to consider the optimal control problem of BSDE. In 2016, Li, Sun and Xiong [11] first studied the stochastic LQ problems under continuous-time MF-BSDEs and established the corresponding fundamental theoretical results. The purpose of this paper is to extend continuous-time mean-field LQ problem

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[†]Corresponding author.

E-mail address: mqx@zjhu.edu.cn(Q. Meng)

to backward jump diffusion system of mean-field type and establish the corresponding theoretical results. We first give notations used throughout our paper and formulate our LQ problem in section 2. In section 3, we prove the existence and uniqueness of the optimal control by classic convex variation principle under standard assumptions. In section 4, we first establish the dual characterization of the optimal control by stochastic Hamiltonian system. Here the stochastic Hamiltonian system turns out to be a full-coupled forward-backward stochastic differential equation of mean-field type with jump, which is very difficult to be solved. Then we introduce two Riccati equations and a MF-BSDE to decouple the stochastic Hamiltonian system. At last, we present explicit formulas of the optimal controls.

2 Notations and Formulation of Problem

2.1 Notations

Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ be a complete probability space on which a d -dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ is defined. Denote by \mathcal{P} the \mathcal{F}_t -predictable σ -field on $[0, T] \times \Omega$ and by $\mathcal{B}(\Lambda)$ the Borel σ -algebra of any topological space Λ . Let $(E, \mathcal{B}(E), v)$ be a measurable space with $v(E) < \infty$ and $\eta : \Omega \times D_\eta \rightarrow E$ be an \mathcal{F}_t -adapted stationary Poisson point process with characteristic measure v , where D_η is a countable subset of $(0, \infty)$. Then the counting measure induced by η is

$$\mu((0, t] \times A) := \#\{s \in D_\eta; s \leq t, \eta(s) \in A\}, \quad \text{for } t > 0, A \in \mathcal{B}(E).$$

And $\tilde{\mu}(de, dt) := \mu(de, dt) - v(de)dt$ is a compensated Poisson random martingale measure which is assumed to be independent of Brownian motion $\{W(t), 0 \leq t \leq T\}$. Assume $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is P -completed natural filtration generated by $\{W(t), 0 \leq t \leq T\}$ and $\{\iint_{A \times (0, t]} \tilde{\mu}(de, ds), 0 \leq t \leq T, A \in \mathcal{B}(E)\}$. In the following, we introduce the basic notations used throughout this paper.

- t : $t \in [0, T)$.
- H : a Hilbert space with norm $\|\cdot\|_H$.
- $\langle \alpha, \beta \rangle$: the inner product in $\mathbb{R}^n, \forall \alpha, \beta \in \mathbb{R}^n$.
- $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm of $\mathbb{R}^n, \forall \alpha \in \mathbb{R}^n$.
- $\langle A, B \rangle = \text{tr}(AB^\top)$: the inner product in $\mathbb{R}^{n \times m}, \forall A, B \in \mathbb{R}^{n \times m}$. Here denote by B^\top the transpose of a matrix

B.

- $|A| = \sqrt{\text{tr}(AA^\top)}$: the norm of A .
- S^n : the set of all $n \times n$ symmetric matrices.
- S_+^n : the subset of all non-negative definite matrices of S^n .
- $S_{\mathcal{F}}^2(t, T; H)$: the space of all H -valued and \mathcal{F}_t -adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in [t, T] \times \Omega\}$

satisfying

$$\|f\|_{S_{\mathcal{F}}^2(t, T; H)}^2 \triangleq \mathbb{E} \left[\sup_{t \leq s \leq T} \|f(s)\|_H^2 ds \right] < +\infty.$$

- $M_{\mathcal{F}}^2(t, T; H)$: the space of all H -valued and \mathcal{F}_s -adapted processes $f = \{f(s, \omega), (s, \omega) \in [0, T] \times \Omega\}$ satisfying

$$\|f\|_{M_{\mathcal{F}}^2(t, T; H)}^2 \triangleq \mathbb{E} \left[\int_t^T \|f(s)\|_H^2 dt \right] < \infty.$$

- $M^{\nu, 2}(E; H)$: the space of all H -valued measurable functions $r = \{r(\theta), \theta \in E\}$ defined on the measure space $(E, \mathcal{B}(E); v)$ satisfying

$$\|r\|_{M^{\nu, 2}(E; H)}^2 \triangleq \int_E \|r(\theta)\|_H^2 v(d\theta) < \infty.$$

- $M_{\mathcal{F}}^{\nu, 2}([t, T] \times E; H)$: the space of all $M^{\nu, 2}(E; H)$ -valued and \mathcal{F}_s -predictable processes $r = \{r(s, \omega, e), (s, \omega, e) \in [t, T] \times \Omega \times E\}$ satisfying

$$\|r\|_{M_{\mathcal{F}}^{\nu, 2}([t, T] \times E; H)}^2 \triangleq \mathbb{E} \left[\int_t^T \|r(s, \cdot)\|_{M^{\nu, 2}(E; H)}^2 ds \right] < \infty.$$

- $L^2(\Omega, \mathcal{F}, P; H)$: the space of all H -valued random variables ξ on (Ω, \mathcal{F}, P) satisfying

$$\|\xi\|_{L^2(\Omega, \mathcal{F}, P; H)} \triangleq \mathbb{E}[\|\xi\|_H^2] < \infty.$$

2.2 Formulation of Problem

Consider the following controlled linear MF-BSDE driven by Brownian motion $\{W(s)\}_{t \leq s \leq T}$ and Poisson random martingale measure $\{\tilde{\mu}(d\theta, ds)\}_{t \leq s \leq T}$

$$\begin{cases} dY(s) = \left\{ A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)] \right. \\ \quad \left. + \int_E D(s, e)R(s, e)\nu(de) + \int_E \bar{D}(s, e)\mathbb{E}[R(s, e)]\nu(de) \right\} dt + Z(s)dW(s) \\ \quad + \int_E R(s, e)\tilde{\mu}(d\theta, ds), s \in [t, T], \\ Y(T) = \xi, \end{cases} \quad (2.1)$$

with the following quadratic cost functional

$$\begin{aligned} J(t, \xi; u(\cdot)) = & \mathbb{E} \left\{ \int_t^T \left(\langle Q(s)Y(s), Y(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[Y(s)], \mathbb{E}[Y(s)] \rangle + \langle N_1(s)Z(s), Z(s) \rangle + \langle \bar{N}_1(s)\mathbb{E}[Z(s)], \mathbb{E}[Z(s)] \rangle \right. \right. \\ & + \int_E \langle N_2(s, e)R(s, e), R(s, e) \rangle \nu(de) + \int_E \langle \bar{N}_2(s, e)\mathbb{E}[R(s, e)], \mathbb{E}[R(s, e)] \rangle \nu(de) \\ & + \langle \bar{N}_3(s)u(s), u(s) \rangle + \langle \bar{N}_3(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \Big) ds \\ & \left. + \langle GY(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right\}, \end{aligned} \quad (2.2)$$

where $A(\cdot), \bar{A}(\cdot), B(\cdot), \bar{B}(\cdot), C(\cdot), \bar{C}(\cdot), D(\cdot, \cdot), \bar{D}(\cdot, \cdot), Q(\cdot), \bar{Q}(\cdot), N_1(\cdot), \bar{N}_1(\cdot), N_2(\cdot, \cdot), \bar{N}_2(\cdot, \cdot), N_3(\cdot), \bar{N}_3(\cdot)$ are given deterministic matrix-valued functions; G and \bar{G} are given matrices; ξ is an \mathcal{F}_T -measurable random variable.

In the above, $u(\cdot)$ is our admissible control process. In this paper, a predictable stochastic process $u(\cdot)$ is said to be an admissible control, if $u(\cdot) \in M_{\mathcal{F}}^2(t, T; \mathbb{R}^m)$. The set of all admissible controls is denoted by $\mathcal{A}[t, T]$. It is obviously that $\mathcal{A}[t, T]$ is a reflexive Banach space whose norm denoted by $\|\cdot\|_{\mathcal{A}[t, T]}$ is defined by

$$\|u(\cdot)\|_{\mathcal{A}[t, T]} = \left\{ \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] \right\}^{\frac{1}{2}}, \forall u(\cdot) \in \mathcal{A}[t, T].$$

For any admissible control $u(\cdot)$, the strong solution of the system (2.1), denoted by $(Y^{(t, \xi, u)}(\cdot), Z^{(t, \xi, u)}(\cdot), R^{(t, \xi, u)}(\cdot))$ or $(Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ if its dependence on admissible control $u(\cdot)$ is clear from the context, is called the state process corresponding to the control process $u(\cdot)$, and $(u(\cdot); Y(\cdot), Z(\cdot), R(\cdot))$ is called an admissible pair.

Our mean-field backward stochastic linear quadratic (LQ) optimal control problem with jump can be stated as follows:

Problem 2.1. For any given $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^n)$, find an admissible control $u^*(\cdot) \in \mathcal{A}[t, T]$ such that

$$J(t, \xi; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}[t, T]} J(t, \xi, u(\cdot)). \quad (2.3)$$

Any $u^*(\cdot) \in \mathcal{A}[t, T]$ satisfying the above is called an optimal control process of Problem 2.1 and the corresponding state process $(Y^*(\cdot), Z^*(\cdot), R^*(\cdot, \cdot))$ is called the optimal state process. Correspondingly $(u^*(\cdot); Y^*(\cdot), Z^*(\cdot), R^*(\cdot, \cdot))$ is called an optimal pair of Problem 2.1.

Throughout this paper, we make the following assumptions on the coefficients.

Assumption 2.1. The matrix-valued functions $A, \bar{A}, C, \bar{C}, Q, \bar{Q}, N_1, \bar{N}_1 : [0, T] \rightarrow \mathbb{R}^{n \times n}; B, \bar{B} : [0, T] \rightarrow \mathbb{R}^{n \times m}; D, \bar{D} : [0, T] \rightarrow \mathcal{L}^{v,2}(E; \mathbb{R}^{n \times n}), N_2, \bar{N}_2 : [0, T] \rightarrow \mathcal{L}^{v,2}(E; \mathbb{R}^{n \times n}); N_3, \bar{N}_3 : [0, T] \rightarrow \mathbb{R}^{m \times m}$ are uniformly bounded measurable functions.

Assumption 2.2. The matrix-valued functions $Q, Q + \bar{Q}, N_1, N_1 + \bar{N}_1, N_2 + \bar{N}_2, N_3, N_3 + \bar{N}_3$ are a.e. nonnegative matrices, and $G, G + \bar{G}$ are nonnegative matrices. Moreover, $N_3, N_3 + \bar{N}_3$ uniformly positive, i.e. for $\forall u \in \mathbb{R}^m$ and a.s. $s \in [t, T]$, $\langle N_3(s)u, u \rangle \geq \delta \langle u, u \rangle$ and $\langle (N_3(s) + \bar{N}_3(s))u, u \rangle \geq \delta \langle u, u \rangle$, for some positive constant δ .

The following result gives the well-posedness of the state equation as well as some useful estimates.

Lemma 2.2. *Let Assumption 2.1 be satisfied. Then for any $(\xi, u(\cdot)) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) \times \mathcal{A}[t, T]$, the state equation (2.1) has a unique solution $\Lambda(\cdot) := (Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2[t, T] := S_{\mathcal{F}}^2(t, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(t, T; \mathbb{R}^n) \times M_{\mathcal{F}}^{\nu,2}([t, T] \times E; \mathbb{R}^n)$. Moreover, we have the following estimate*

$$\|\Lambda(\cdot)\|_{M^2[t, T]}^2 \leq K \left\{ \|u(\cdot)\|_{\mathcal{A}[t, T]}^2 + E[\|\xi\|^2] \right\} \quad (2.4)$$

and

$$|J(t, \xi; u(\cdot))| < \infty. \quad (2.5)$$

Suppose that $\bar{\Lambda}(\cdot) := (\bar{Y}(\cdot), \bar{Z}(\cdot), \bar{R}(\cdot, \cdot))$ is the solution to the state equation (2.1) corresponding to another $(\bar{\xi}, \bar{u}(\cdot)) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n) \times \mathcal{A}[t, T]$, then we have the following estimate

$$\|\Lambda(\cdot) - \bar{\Lambda}(\cdot)\|_{M^2[t, T]}^2 \leq K \left\{ \|u(\cdot) - \bar{u}(\cdot)\|_{\mathcal{A}[t, T]}^2 + E[\|\xi - \bar{\xi}\|^2] \right\}. \quad (2.6)$$

Here we define

$$\|\Lambda(\cdot)\|_{M^2[t, T]}^2 =: \mathbb{E} \left[\sup_{t \leq s \leq T} |Y(s)|^2 \right] + \mathbb{E} \left[\int_t^T |Z(s)|^2 ds \right] + \mathbb{E} \left[\int_t^T \int_E |R(s, e)|^2 \nu(de) ds \right].$$

Proof. The existence and uniqueness of the solution can be obtained by a standard argument using the contraction mapping theorem. For the estimates (2.4) and (2.6), we can easily obtain them by applying the Itô formula to $|Y(\cdot)|^2$ and $|Y(\cdot) - \bar{Y}(\cdot)|^2$, Gronwall inequality and B-D-G inequality. For the estimate (2.5), using Assumption 2.1 and the estimate (2.4), we have

$$|J(t, \xi; u(\cdot))| \leq K \left\{ \|\Lambda(\cdot)\|_{M^2[t, T]}^2 + \|u(\cdot)\|_{\mathcal{A}[t, T]}^2 \right\} \leq K \left\{ \|u(\cdot)\|_{\mathcal{A}[t, T]}^2 + E[\|\xi\|^2] \right\} < \infty, \quad (2.7)$$

where we have used the elementary inequality: for any $\Phi \in L^2(\Omega, \mathcal{F}, P; H)$,

$$\|\mathbb{E}[\Phi]\|_H^2 \leq \mathbb{E}\|\Phi\|_H^2. \quad (2.8)$$

The proof is complete. \square

Therefore, by Lemma 2.2, we know that Problem 2.1 is well-defined.

3 Existence and Uniqueness of Optimal Control

In this section, we study the existence and uniqueness of the optimal control of Problem 2.1. To this end, we first establish some elementary properties of the cost functional.

Lemma 3.1. *Let Assumptions 2.1 and 2.2 be satisfied. Then for any $(t, \xi) \in [0, T) \times L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, the cost functional $J(t, \xi; u(\cdot))$ is continuous over $\mathcal{A}[t, T]$.*

Proof. Suppose that $(u(\cdot); \Lambda(\cdot)) = (u(\cdot); Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ and $(\bar{u}(\cdot); \bar{\Lambda}(\cdot)) = (\bar{u}(\cdot); \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{R}(\cdot, \cdot))$ be any two admissible control pairs. Under Assumptions 2.1 and 2.2, from the definition of the cost functional $J(t, \xi; u(\cdot))$ (see (2.2)), we have

$$\begin{aligned} & |J(t, \xi; u(\cdot)) - J(t, \xi; \bar{u}(\cdot))|^2 \\ & \leq K \left\{ \|\Lambda(\cdot) - \bar{\Lambda}(\cdot)\|_{M^2[t, T]}^2 + \|u(\cdot) - \bar{u}(\cdot)\|_{\mathcal{A}[t, T]}^2 \right\} \times \left\{ \|\Lambda(\cdot)\|_{M^2[t, T]}^2 + \|u(\cdot)\|_{\mathcal{A}[t, T]}^2 + \|\bar{\Lambda}(\cdot)\|_{M^2[t, T]}^2 + \|\bar{u}(\cdot)\|_{\mathcal{A}[t, T]}^2 \right\}. \end{aligned} \quad (3.1)$$

Using the estimates (2.4) and (2.6) lead to

$$|J(t, \xi; u(\cdot)) - J(t, \xi; \bar{u}(\cdot))|^2 \leq K \left\{ \|u(\cdot) - \bar{u}(\cdot)\|_{\mathcal{A}[t, T]}^2 \right\} \times \left\{ \|u(\cdot)\|_{\mathcal{A}[t, T]}^2 + \|\bar{u}(\cdot)\|_{\mathcal{A}[t, T]}^2 + \mathbb{E}[\|\xi\|^2] \right\}. \quad (3.2)$$

Thus, we get that

$$J(t, \xi; u(\cdot)) - J(t, \xi; \bar{u}(\cdot)) \rightarrow 0, \quad \text{as } u(\cdot) \rightarrow \bar{u}(\cdot) \quad \text{in } \mathcal{A}[t, T]. \quad (3.3)$$

The proof is complete. \square

Lemma 3.2. *Let Assumptions 2.1 and 2.2 be satisfied. Then for any given $(t, \xi) \in [0, T] \times L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, the cost functional $J(t, \xi; u(\cdot))$ is strictly convex $\mathcal{A}[0, T]$. Moreover, the cost functional $J(t, \xi; u(\cdot))$ is coercive over $\mathcal{A}[0, T]$, i.e.,*

$$\lim_{\|u(\cdot)\|_{\mathcal{A}[t, T]} \rightarrow \infty} J(t, \xi; u(\cdot)) = \infty.$$

Proof. Since the weighting matrices in the cost functional is not random, it is easy to check that

$$\begin{aligned} J(t, \xi; u(\cdot)) &= \mathbb{E} \left[\int_t^T \left(\langle Q(s)(Y(s) - \mathbb{E}[Y(s)]), Y(s) - \mathbb{E}[Y(s)] \rangle + \langle (Q + \bar{Q})(s) \mathbb{E}[Y(s)], \mathbb{E}[Y(s)] \rangle \right. \right. \\ &\quad + \langle N_1(s)(Z(s) - \mathbb{E}[Z(s)]), Z(s) - \mathbb{E}[Z(s)] \rangle + \langle (N_1(s) + \bar{N}_1(s)) \mathbb{E}[Z(s)], \mathbb{E}[Z(s)] \rangle \\ &\quad + \int_E \langle N_2(s, e)(R(s, e) - \mathbb{E}[R(s, e)]), R(s, e) - \mathbb{E}[R(s, e)] \rangle \nu(de) \\ &\quad + \int_E \langle (N_2(s, e) + \bar{N}_2(s, e)) \mathbb{E}[R(s, e)], \mathbb{E}[R(s, e)] \rangle \nu(de) \\ &\quad + \langle N_3(s)(u(s) - \mathbb{E}[u(s)]), u(s) - \mathbb{E}[u(s)] \rangle + \langle (N_3(s) + \bar{N}_3(s)) \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \Big) ds \Big] \\ &\quad + \mathbb{E} \left[\langle G(Y(t) - \mathbb{E}[Y(t)]), Y(t) - \mathbb{E}[Y(t)] \rangle + \langle (G + \bar{G}) \mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right]. \end{aligned} \quad (3.4)$$

Thus the cost functional $J(t, \xi; u(\cdot))$ over $\mathcal{A}[t, T]$ is convex from the nonnegativity of the $N_1, N_1 + \bar{N}_1, N_2, N_2 + \bar{N}_2, N_3 + \bar{N}_3, Q, Q + \bar{Q}, G, G + \bar{G}$. Actually, since $N_3, N_3 + \bar{N}_3$ is uniformly positive, $J(t, \xi; u(\cdot))$ is strictly convex. On the other hand, by Assumption 2.2 and (3.4), we get

$$\begin{aligned} J(t, \xi; u(\cdot)) &\geq \mathbb{E} \left[\int_t^T \left(\langle N_3(s)(u(s) - \mathbb{E}[u(s)]), u(s) - \mathbb{E}[u(s)] \rangle + \langle (N_3(s) + \bar{N}_3(s)) \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right) ds \right] \\ &\geq \delta \mathbb{E} \left[\int_t^T \langle u(s) - \mathbb{E}[u(s)], u(s) - \mathbb{E}[u(s)] \rangle ds \right] + \delta \mathbb{E} \left[\int_t^T \langle \mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle ds \right] \\ &= \delta \mathbb{E} \left[\int_t^T |u(s)|^2 ds \right] \\ &= \delta \|u(\cdot)\|_{\mathcal{A}[t, T]}^2. \end{aligned} \quad (3.5)$$

Thus $\lim_{\|u(\cdot)\|_{\mathcal{A}[t, T]} \rightarrow \infty} J(t, \xi; u(\cdot)) = \infty$. The proof is complete. \square

Lemma 3.3. *Let Assumptions 2.1 and 2.2 be satisfied. Then for any given $(t, \xi) \in [0, T] \times L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, the cost functional $J(t, \xi; u(\cdot))$ is Fréchet differentiable over $\mathcal{A}[t, T]$ and the corresponding Fréchet derivative $J'(t, \xi; u(\cdot))$ is given by*

$$\begin{aligned}
\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = & 2\mathbb{E} \left[\int_t^T \left(\langle Q(s)Y^{(t, \xi, u)}(s), Y^{(t, 0, v)}(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[Y^{(t, \xi, u)}(s)], \mathbb{E}[Y^{(t, 0, v)}(s)] \rangle \right. \right. \\
& + \langle N_1(s)Z^{(t, \xi, u)}(s), Z^{(t, 0, v)}(s) \rangle + \langle \bar{N}_1(s)\mathbb{E}[Z^{(t, \xi, u)}(s)], \mathbb{E}[Z^{(t, 0, v)}(s)] \rangle \\
& + \int_E \langle N_2(s, e)R^{(t, \xi, u)}(s, e), R^{(t, 0, v)}(s, e) \rangle \nu(de) \\
& + \int_E \langle \bar{N}_2(s)\mathbb{E}[R^{(t, \xi, u)}(s, e)], \mathbb{E}[R^{(t, 0, v)}(s, e)] \rangle \nu(de) \\
& \left. \left. + \langle N_3(s)u(s), v(s) \rangle + \langle \bar{N}_3(s)\mathbb{E}[u(s)], \mathbb{E}[v(s)] \rangle \right) ds \right] \\
& + 2\mathbb{E} \left[\langle GY^{(t, \xi, u)}(t), Y^{(t, 0, v)}(t) \rangle + \langle \bar{G}\mathbb{E}[Y^{(t, \xi, u)}(t)], \mathbb{E}[Y^{(t, 0, v)}(t)] \rangle \right], \quad \forall u(\cdot), v(\cdot) \in \mathcal{A}[t, T],
\end{aligned} \tag{3.6}$$

where $(Y^{(t, 0, v)}(\cdot), Z^{(t, 0, v)}(\cdot), R^{(t, 0, v)}(\cdot, \cdot))$ is the solution of the following MF-BSDE

$$\begin{cases} dY(s) = \left\{ A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)v(s) + \bar{B}(s)\mathbb{E}[v(s)] + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)] \right. \\ \quad \left. + \int_E D(s, e)R(s, e)\nu(de) + \int_E \bar{D}(s, e)\mathbb{E}[R(s, e)]\nu(de) \right\} ds + Z(s)dW(s) \\ \quad + \int_E R(s, e)\tilde{\mu}(d\theta, ds), s \in [t, T], \\ Y(T) = 0. \end{cases} \tag{3.7}$$

Proof. Let $u(\cdot)$ and $v(\cdot)$ be two any given admissible control. For simplicity, the right hand side of (3.6) is denoted by $\Delta^{u, v}$. Since the state equation (2.1) is linear, by the uniqueness of the solution of the MF-BSDE, it is easily to check that

$$\begin{aligned}
Y^{(t, \xi, u+v)}(s) &= Y^{(s, \xi, u)}(s) + Y^{(t, 0, v)}(s), \\
Z^{(t, \xi, u+v)}(s) &= Z^{(t, \xi, u)}(s) + Z^{(t, 0, v)}(s), \\
R^{(t, \xi, u+v)}(s) &= R^{(t, \xi, u)}(s) + R^{(t, 0, v)}(s), t \leq s \leq T.
\end{aligned} \tag{3.8}$$

Therefore, in terms of (3.8) and the definition of the cost functional $J(x, u(\cdot))$ (see (2.2)), it is easy to check that

$$J(t, \xi; u(\cdot) + v(\cdot)) - J(t, \xi; u(\cdot)) = J(t, 0; v(\cdot)) + \Delta^{u, v}. \tag{3.9}$$

On the other hand, the estimate (2.7) leads to

$$|J(t, 0; v(\cdot))| \leq K \|v(\cdot)\|_{\mathcal{A}[t, T]}^2. \tag{3.10}$$

Therefore,

$$\lim_{\|v(\cdot)\|_{\mathcal{A}[t, T]} \rightarrow 0} \frac{|J(t, \xi; u(\cdot) + v(\cdot)) - J(t, \xi; u(\cdot)) - \Delta^{u, v}|}{\|v(\cdot)\|_{\mathcal{A}[t, T]}} = \lim_{\|v(\cdot)\|_{\mathcal{A}[t, T]} \rightarrow 0} \frac{|J(t, 0; v(\cdot))|}{\|v(\cdot)\|_{\mathcal{A}[t, T]}} = 0, \tag{3.11}$$

which gives that $J(t, \xi; u(\cdot))$ has Fréchet derivative $\Delta^{u, v}$. The proof is complete. \square

Remark 3.1. Since the cost functional $J(t, \xi; u(\cdot))$ is Fréchet differentiable, then it is also Gâteaux differentiable.

Moreover, the Gâteaux derivative is the Fréchet derivative $\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle$. In fact, from (3.9), we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{J(t, \xi; u(\cdot) + \varepsilon v(\cdot)) - J(t, \xi; u(\cdot))}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{J(t, 0; \varepsilon v(\cdot)) + \Delta^{u, \varepsilon v}}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2 J(t, 0; v(\cdot)) + \varepsilon \Delta^{u, v}}{\varepsilon} \\
&= \Delta^{u, v} \\
&= \langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle.
\end{aligned} \tag{3.12}$$

Now by Lemma 3.1-3.3, we can obtain the existence and uniqueness of optimal control. This result is stated as follows.

Theorem 3.4. *Let Assumptions 2.1 and 2.2 be satisfied. Then Problem 2.1 has a unique optimal control.*

Proof. Since the admissible controls set $\mathcal{A}[t, T] = M_{\mathcal{F}}^2(t, T; \mathbb{R}^m)$ is a reflexive Banach space, in terms of Lemma 3.1-3.3, the uniqueness and existence of the optimal control of Problem 2.1 can be directly got from Proposition 2.12 of [7] (i.e., the coercive, strictly convex and lower-semi continuous functional defined on the reflexive Banach space has a unique minimum value point). The proof is complete. \square

Theorem 3.5. *Let Assumptions 2.1 and 2.2 be satisfied. Then a necessary and sufficient condition for an admissible control $u(\cdot) \in \mathcal{A}[t, T]$ to be an optimal control of Problem 2.1 is that for any admissible control $v(\cdot) \in \mathcal{A}[t, T]$,*

$$\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0, \tag{3.13}$$

i.e.

$$\begin{aligned}
0 = & 2\mathbb{E} \left[\int_t^T \left(\langle Q(s)Y^{(t, \xi, u)}(s), Y^{(t, 0, v)}(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[Y^{(x, u)}(s)], \mathbb{E}[Y^{(t, 0, v)}(s)] \rangle \right. \right. \\
& + \langle N_1(s)Z^{(t, \xi, u)}(s), Z^{(t, 0, v)}(s) \rangle + \langle \bar{N}_1(s)\mathbb{E}[Z^{(t, \xi, u)}(s)], \mathbb{E}[Z^{(t, 0, v)}(s)] \rangle \\
& + \int_E \langle N_2(s, e)R^{(t, \xi, u)}(s, e), R^{(t, 0, v)}(s, e) \rangle \nu(de) \\
& + \int_E \langle \bar{N}_2(s)\mathbb{E}[R^{(t, \xi, u)}(s, e)], \mathbb{E}[R^{(t, 0, \xi)}(s)] \rangle \nu(de) \\
& \left. + \langle N_3(s)u(s), v(s) \rangle + \langle \bar{N}_3(s)\mathbb{E}[u(s)], \mathbb{E}[v(s)] \rangle \right) ds \Big] \\
& + 2\mathbb{E} \left[\langle GY^{(t, \xi, u)}(t), Y^{(t, 0, v)}(t) \rangle + \langle \bar{G}\mathbb{E}[Y^{(t, \xi, u)}(t)], \mathbb{E}[Y^{(t, 0, v)}(t)] \rangle \right], \quad \forall u(\cdot), v(\cdot) \in \mathcal{A}[t, T].
\end{aligned} \tag{3.14}$$

Proof. For the necessary part, suppose that $u(\cdot)$ is an optimal control. Then from (3.12), for any admissible control $v(\cdot)$ and $0 < \varepsilon \leq 1$, we have

$$\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{J(t, \xi; u(\cdot) + \varepsilon v(\cdot)) - J(t, \xi; u(\cdot))}{\varepsilon} \geq 0, \tag{3.15}$$

and

$$-\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = \langle J'(t, \xi; u(\cdot)), -v(\cdot) \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{J(t, \xi; u(\cdot) + \varepsilon(-v(\cdot))) - J(t, \xi; u(\cdot))}{\varepsilon} \geq 0, \tag{3.16}$$

which imply that

$$\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0. \tag{3.17}$$

For the sufficient part, let $u(\cdot)$ be an given admissible control, and suppose that for any admissible control $v(\cdot)$, $\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0$. Since the cost functional J is convex, then we have

$$J(t, \xi; v(\cdot)) - J(t, \xi; u(\cdot)) \geq \langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0, \tag{3.18}$$

which implies that $u(\cdot)$ is an optimal control. The proof is complete. \square

4 Stochastic Hamilton Systems, decoupling, and Riccati equations, Representations of optimal controls

4.1 Stochastic Hamilton Systems

In this subsection, we give the characterization of optimal control of Problem 2.1 by the stochastic Hamilton system. To simplify our notation, in what follows, we shall often suppress the time variable s if no confusion can arise.

Theorem 4.1. *Let Assumptions 2.1 and 2.2 be satisfied. Then, a necessary and sufficient condition for an admissible pair $(u(\cdot); Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ to be an optimal pair of Problem 2.1 is that the admissible control $u(\cdot)$ satisfies*

$$N_3(s)u(s) + \bar{N}_3(s)\mathbb{E}[u(s)] + B^\top(s)k(s-) + \bar{B}^\top(s)\mathbb{E}[k(s-)] = 0, \quad a.e.a.s., s \in [t, T], \quad (4.1)$$

where $k(\cdot)$ is the unique solution of the following MF-SDE

$$\begin{cases} dk(s) = - \left[A^\top(s)k(s) + \bar{A}(s)^\top \mathbb{E}[k(s)] + Q(s)Y(s) + \bar{Q}(s)\mathbb{E}[Y(s)] \right] ds \\ \quad - \left[C^\top(s)k(s) + \bar{C}(s)^\top \mathbb{E}[k(s)] + 2N_1(s)Z(s) + \bar{N}_1(s)\mathbb{E}[Z(s)] \right] dW(s) \\ \quad - \int_E \left[D^\top(s, e)k(s) + \bar{D}(s, e)^\top \mathbb{E}[k(s)] + N_2(s, e)R(s, e) + \bar{N}_2(s, e)\mathbb{E}[R(s, e)] \right] \tilde{\mu}(d\theta, ds), s \in [t, T], \\ k(t) = -GY(t) - \bar{G}\mathbb{E}[Y(t)]. \end{cases} \quad (4.2)$$

Proof. Let $u(\cdot) \in \mathcal{A}[t, T]$ be a given admissible control. Then for any admissible control $v(\cdot) \in \mathcal{A}[t, T]$, from Lemma 3.3, we have

$$\begin{aligned} & \langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle \\ &= 2\mathbb{E} \left[\int_t^T \left(\langle QY^{(t, \xi, u)}, Y^{(t, 0, v)} \rangle + \langle \bar{Q}\mathbb{E}[Y^{(x, u)}], \mathbb{E}[Y^{(t, 0, v)}] \rangle + \langle N_1Z^{(t, \xi, u)}, Z^{(t, 0, v)} \rangle \right. \right. \\ & \quad \left. \left. + \langle \bar{N}_1\mathbb{E}[Z^{(t, \xi, u)}], \mathbb{E}[Z^{(t, 0, v)}] \rangle + \int_E \langle N_2R^{(t, \xi, u)}, R^{(t, 0, v)} \rangle \nu(de) \right. \right. \\ & \quad \left. \left. + \int_E \langle \bar{N}_2\mathbb{E}[R^{(t, \xi, u)}], \mathbb{E}[R^{(t, 0, \xi)}] \rangle \nu(de) + \langle N_3u, v \rangle + \langle \bar{N}_3\mathbb{E}[u], \mathbb{E}[v] \rangle \right) ds \right] \\ & \quad + 2\mathbb{E} \left[\langle GY^{(t, \xi, u)}(t), Y^{(t, 0, v)}(t) \rangle + \langle \bar{G}\mathbb{E}[Y^{(t, \xi, u)}(t)], \mathbb{E}[Y^{(t, 0, v)}(t)] \rangle \right]. \end{aligned} \quad (4.3)$$

On the other hand, by [23], we know that (4.2) admits a unique adapted solution $k(\cdot)$. Applying Itô's formula to $\langle Y^{t, 0; u}(s), k(s) \rangle$ and taking expectation, we have

$$\begin{aligned} & \mathbb{E} \left[\int_t^T \left(\langle QY^{(t, \xi, u)}, Y^{(t, 0, v)} \rangle + \langle \bar{Q}\mathbb{E}[Y^{(x, u)}], \mathbb{E}[Y^{(t, 0, v)}] \rangle + \langle N_1Z^{(t, \xi, u)}, Z^{(t, 0, v)} \rangle + \langle \bar{N}_1\mathbb{E}[Z^{(t, \xi, u)}], \mathbb{E}[Z^{(t, 0, v)}] \rangle \right. \right. \\ & \quad \left. \left. + \int_E \langle N_2R^{(t, \xi, u)}, R^{(t, 0, v)} \rangle \nu(de) + \int_E \langle \bar{N}_2\mathbb{E}[R^{(t, \xi, u)}], \mathbb{E}[R^{(t, 0, \xi)}] \rangle \nu(de) \right) ds \right] \\ & \quad + \mathbb{E} \left[\langle GY^{(t, \xi, u)}(t), Y^{(t, 0, v)}(t) \rangle + \langle \bar{G}\mathbb{E}[Y^{(t, \xi, u)}(t)], \mathbb{E}[Y^{(t, 0, v)}(t)] \rangle \right] \\ &= \mathbb{E} \left[\int_t^T \left(\langle k, Bv + \bar{B}\mathbb{E}[v] \rangle \right) ds \right] \\ &= \mathbb{E} \left[\int_t^T \left\langle B^\top k + \bar{B}^\top \mathbb{E}[k], v \right\rangle ds \right]. \end{aligned} \quad (4.4)$$

Putting (4.4) into (4.3), we get

$$\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 2\mathbb{E} \left[\int_t^T \left\langle N_3(s)u(s) + \bar{N}_3(s)\mathbb{E}[u(s)] + B^\top(s)k(s-) + \bar{B}^\top(s)\mathbb{E}[k(s-)], v(s) \right\rangle ds \right]. \quad (4.5)$$

For the necessary, let $(u(\cdot); Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ be an optimal pair, then from Theorem 3.5, we have $\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0$ which imply that

$$N_3(s)u(s) + \bar{N}_3(s)\mathbb{E}[u(s)] + B^\top(s)k(s-) + \bar{B}^\top(s)\mathbb{E}[k(s-)] = 0, \quad a.e.a.s., s \in [t, T], \quad (4.6)$$

from (4.5), since $v(\cdot)$ is arbitrary.

For the sufficient part, let $(u(\cdot); Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ be an admissible pair satisfying (4.1). Putting (4.1) into (4.5), then we have $\langle J'(t, \xi; u(\cdot)), v(\cdot) \rangle = 0$, which implies that $(u(\cdot); Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ is an optimal control pair from Theorem 3.5. \square

Finally, we introduce the so-called stochastic Hamilton system which consists of the state equation (2.1), the adjoint equation (4.2) and the dual representation (4.1):

$$\left\{ \begin{array}{l} dY(s) = \left\{ A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)] \right. \\ \quad \left. - \int_E D(s, e)R(s, e)\nu(de) + \int_E \bar{D}(s, e)\mathbb{E}[R(s, e)]\nu(de) \right\} ds + Z(s)dW(s) + \int_E R(t, e)\tilde{\mu}(d\theta, ds), \\ dk(s) = - \left[A(s)^\top k(s) + \bar{A}(s)^\top \mathbb{E}[k(s)] + Q(s)Y(s) + \bar{Q}(s)\mathbb{E}[Y(s)] \right] ds \\ \quad - \left[C(s)^\top k(s) + \bar{C}(s)^\top \mathbb{E}[k(s)] + N_1(s)Z(s) + \bar{N}_1(s)\mathbb{E}[Z(s)] \right] dW(s) \\ \quad - \int_E \left[D(s, e)^\top k(s) + \bar{D}(s, e)^\top \mathbb{E}[k(s)] + N_2(s, e)R(s, e) + \bar{N}_2(s, e)\mathbb{E}[R(s, e)] \right] \tilde{\mu}(d\theta, ds), \\ Y(T) = \xi, k(t) = -GY(t) - \bar{G}\mathbb{E}[Y(t)], \\ N_3(s)u(s) + \bar{N}_3(s)\mathbb{E}[u(s)] + B^\top(s)k(t-) + \bar{B}^\top(s)\mathbb{E}[k(t-)] = 0, \quad s \in [t, T]. \end{array} \right. \quad (4.7)$$

This is a fully coupled mean-field forward- backward stochastic differential equation (MF-FBSDE in short) with jump and its solution consists of $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), K(\cdot))$.

Theorem 4.2. *Let Assumptions 2.1 and 2.2 be satisfied. Then stochastic Hamilton system (4.7) has a unique solution $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot)) \in M_{\mathcal{F}}^2(t, T; \mathbb{R}^m) \times S_{\mathcal{F}}^2(t, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(t, T; \mathbb{R}^n) \times M_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathbb{R}^n) \times S_{\mathcal{F}}^2(t, T; \mathbb{R}^n)$. Moreover $u(\cdot)$ is the unique optimal control of Problem 2.1 and $(Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ is its corresponding optimal state process.*

Proof. By Theorem 3.4, Problem 2.1 admits a unique optimal pair $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot))$. Suppose $k(\cdot)$ is the unique solution of the adjoint equation (4.2) corresponding to the optimal pair $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot))$. Then by the necessary part of Theorem 4.1, the optimal control has the dual presentation (4.1). Consequently, $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot))$ consists of an adapted solution to the stochastic Hamilton system (4.7). Next, if the stochastic Hamilton system (4.7) has an another adapted solution $(\bar{u}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{R}(\cdot, \cdot), \bar{k}(\cdot))$, then $(\bar{u}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{R}(\cdot, \cdot))$ must be an optimal pair of Problem 2.1 by the sufficient part of Theorem 4.1. So we must have $u(\cdot) = \bar{u}(\cdot)$ by the uniqueness of the optimal control. Furthermore, by the uniqueness of the solutions of MF-SDE and MF-BSDE, one must have $(\bar{u}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{R}(\cdot, \cdot), \bar{k}(\cdot)) = (u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot))$. The proof is complete. \square

In summary, the stochastic Hamilton system (4.7) completely characterizes the optimal control of Problem 2.1. Therefore, solving Problem 2.1 is equivalent to solving the stochastic Hamilton system, moreover, the unique optimal control can be given by (4.1). Taking expectation to (4.1), we have

$$N_3(t)u(t) + \bar{N}_3(t)\mathbb{E}[u(t)] + B^\top(t)k(t-) + \bar{B}^\top(t)\mathbb{E}[k(t-)] = 0, \quad a.e.a.s., \quad (4.8)$$

which implies that

$$\mathbb{E}[u(t)] = -(N_3(t) + \bar{N}_3(t))^{-1} \left[(B(t) + \bar{B}(t))^\top \mathbb{E}[k(t-)] \right], \quad a.e.a.s. \quad (4.9)$$

From (4.1), we know that

$$N_3(s)u(s) = -\bar{N}_3(s)\mathbb{E}[u(s)] - B^\top(s)k(s-) - \bar{B}^\top(s)\mathbb{E}[k(s-)], \quad a.e.a.s.. \quad (4.10)$$

Then putting (4.9) into (4.10), we have

$$\begin{aligned} u(s) = & -N_3^{-1}(s) \left\{ B^\top(s)k(s-) + \bar{B}^\top(s)\mathbb{E}[k(s-)] \right\} \\ & + \bar{N}_3(s)(N_3(s) + \bar{N}_3(s))^{-1} \left[(B(s) + \bar{B}(s))^\top \mathbb{E}[k(s-)] \right], \quad a.e.a.s., \quad \in [t, T]. \end{aligned} \quad (4.11)$$

4.2 Derivation of Riccati equations

Although the optimal control of Problem 2.1 is completely characterized by the stochastic Hamilton system (4.7), (4.7) is a fully coupled mean-field forward-backward stochastic differential equation whose solvability is much difficult to be obtained. To solve the stochastic Hamilton system (4.7), as in [11], we will use the decoupling technique for general FBSDEs which will lead to a derivation of two Riccati equations

Let $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot))$ be the solution of the stochastic Hamilton system (4.7). Taking expectation on both sides of the stochastic Hamilton system (4.7), we get that $(\mathbb{E}[u(\cdot)], \mathbb{E}[Y(\cdot)], \mathbb{E}[Z(\cdot)], \mathbb{E}[R(\cdot, \cdot)], \mathbb{E}[k(\cdot)])$ satisfies the following forward-backward ordinary differential equation

$$\begin{cases} d\mathbb{E}[k(s)] &= - \left[(A^\top(s) + \bar{A}(s)^\top) \mathbb{E}[k(s)] + (Q(s) + \bar{Q}(s)) \mathbb{E}[Y(s)] \right], \\ d\mathbb{E}[Y(s)] &= \left\{ (A(s) + \bar{A}(s)) \mathbb{E}[Y(s)] + (B(s) + \bar{B}(s)) \mathbb{E}[u(s)] + (C(s) + \bar{C}(s)) \mathbb{E}[Z(s)] \right. \\ &\quad \left. + \int_E (D(s, e) + \bar{D}(s, e)) \mathbb{E}[R(s, e)] \nu(de) \right\} ds, \\ \mathbb{E}[Y(T)] &= \mathbb{E}[\xi], \mathbb{E}[k(t)] = -(G + \bar{G}) \mathbb{E}[Y(t)], \\ 0 &= (N_3(s) + \bar{N}_3(s)) \mathbb{E}[u(s)] + (B^\top(s) + \bar{B}^\top(s)) \mathbb{E}[k(s-)]. \end{cases} \quad (4.12)$$

Further, it is easy to check that $(u(\cdot) - \mathbb{E}[u(\cdot)], Y(\cdot) - \mathbb{E}[Y(\cdot)], Z(\cdot) - \mathbb{E}[Z(\cdot)], R(\cdot, \cdot) - \mathbb{E}[R(\cdot, \cdot)], k(\cdot) - \mathbb{E}[k(\cdot)])$ satisfies the following forward-backward stochastic differential equation

$$\begin{cases} dY - \mathbb{E}[Y] &= \left\{ A(Y - \mathbb{E}[Y]) + B(u - \mathbb{E}[u]) + C(Z - \mathbb{E}[Z]) + \int_E D(R - \mathbb{E}[R]) \nu(de) \right\} ds + Z dW + \int_E R \tilde{\mu}(d\theta, ds), \\ dk - \mathbb{E}[k] &= - \left[A^\top(k - \mathbb{E}[k]) + Q(Y - \mathbb{E}[Y]) \right] ds + \left[C^\top(k - \mathbb{E}[k]) - (C^\top + \bar{C}^\top) \mathbb{E}[k] + N_1(Z - \mathbb{E}[Z]) \right. \\ &\quad \left. + (N_1 + \bar{N}_1) \mathbb{E}[Z] \right] dW - \int_E \left[D^\top(k - \mathbb{E}[k]) + (D^\top + \bar{D}^\top) \mathbb{E}[k] + N_2(R - \mathbb{E}[R]) \right. \\ &\quad \left. + (N_2 + \bar{N}_2) \mathbb{E}[R] \right] \tilde{\mu}(d\theta, ds), \\ k(t) - \mathbb{E}[k(t)] &= -G(Y(t) - \mathbb{E}[Y(t)]), Y(T) - \mathbb{E}[Y(T)] = \xi - \mathbb{E}[\xi], \\ 0 &= N_3(u - \mathbb{E}[u]) + B^\top(k(t-) - \mathbb{E}[k(t-)]). \end{cases} \quad (4.13)$$

Now we assume that the state process $Y(\cdot)$ and the adjoint process $k(\cdot)$ have the following relationship:

$$Y(s) = P(s)(k(s) - \mathbb{E}[k(s)]) + \Pi(s)\mathbb{E}[k(s)] + \varphi(s), \quad (4.14)$$

where $P(\cdot), \Pi(\cdot) : [0, T] \rightarrow \mathbb{R}^{n \times n}$ are absolutely continuous and $\varphi(\cdot)$ satisfies the following BSDE with jump

$$d\varphi(s) = \alpha(s)ds + \beta(s)dW(s) + \Phi(s, e)d\tilde{\mu}(de, ds), \varphi(T) = \xi, \quad (4.15)$$

for some \mathbb{F} -progressively measurable process α, β and Φ . Consequently, we further have the following relationship

$$\mathbb{E}[Y(s)] = \Pi(s)\mathbb{E}[k(s)] + \mathbb{E}[\varphi(s)] \quad (4.16)$$

and

$$Y(s) - \mathbb{E}[Y(s)] = P(s)(k(s) - \mathbb{E}[k(s)]) + (\varphi(s) - \mathbb{E}[\varphi(s)]). \quad (4.17)$$

Denote

$$d\eta(s) = \gamma ds + \beta(s)dW(s) + \Phi(s, e)d\tilde{\mu}(de, ds), \eta(T) = \xi - \mathbb{E}[\xi], \quad (4.18)$$

where

$$\eta(s) = \varphi(s) - \mathbb{E}[\varphi(s)], \gamma(s) = \alpha(s) - \mathbb{E}[\alpha(s)].$$

In the following, we begin to formerly derive the corresponding Riccati equations which $P(\cdot)$ and $\Pi(\cdot)$ should satisfy. From the relationships (4.17) and (4.13), applying Itô formula to $P(s)(Y(s) - \mathbb{E}[Y(s)])$ leads to

$$\begin{aligned} & \left\{ A(Y - \mathbb{E}[Y]) + B(u - \mathbb{E}[u]) + C(Z - \mathbb{E}[Z]) + \int_E D(R - \mathbb{E}[R])\nu(de) \right\} ds + ZdW + \int_E R\tilde{\mu}(d\theta, ds) \\ &= d(Y - \mathbb{E}[Y]) \\ &= dP(k - \mathbb{E}[k]) + d\eta \\ &= \left[\dot{P}(k - \mathbb{E}[k]) - PA^\top(k - \mathbb{E}[k]) - PQ(Y - \mathbb{E}[Y]) \right] ds - P \left[C^\top(k - \mathbb{E}[k]) + (C^\top + \bar{C}^\top)\mathbb{E}[k] \right. \\ & \quad \left. + N_1(Z - \mathbb{E}[Z]) + (N_1 + \bar{N}_1)\mathbb{E}[Z] \right] dW - \int_E P \left[D^\top(k - \mathbb{E}[k]) + (D^\top + \bar{D}^\top)\mathbb{E}[k] \right. \\ & \quad \left. + N_2(R - \mathbb{E}[R]) + (N_2 + \bar{N}_2)\mathbb{E}[R] \right] \tilde{\mu}(d\theta, ds) + \gamma ds + \beta dW + \Phi d\tilde{\mu}(de, ds). \end{aligned} \quad (4.19)$$

Comparing the diffusion terms of both sides of the above equality, we have

$$Z = -P \left[C^\top k + \bar{C}^\top \mathbb{E}[k] + N_1 Z + \bar{N}_1 \mathbb{E}[Z] \right] + \beta, \quad (4.20)$$

$$R = -P \left[D^\top k + \bar{D}^\top \mathbb{E}[k] + N_2 R + \bar{N}_2 \mathbb{E}[R] \right] + \Phi, \quad (4.21)$$

and

$$\begin{aligned} & AP(k - \mathbb{E}[k]) + A\eta - BN_3^{-1}B^\top(k - \mathbb{E}[k]) + C(Z - \mathbb{E}[Z]) + \int_E D(R - \mathbb{E}[R])\nu(de) \\ &= \dot{P}(k - \mathbb{E}[k]) - PA^\top(k - \mathbb{E}[k]) - PQP(k - \mathbb{E}[k]) - PQ\eta + \gamma, \end{aligned} \quad (4.22)$$

which imply that

$$\begin{aligned} & \left(\dot{P} - PA^\top - AP - PQP + BN_3^{-1}B^\top \right) (k - \mathbb{E}[k]) \\ & - C(Z - \mathbb{E}[Z]) - \int_E D(R - \mathbb{E}[R])\nu(de) - (A + PQ)\eta(t) + \gamma = 0. \end{aligned} \quad (4.23)$$

Then taking expectation on both sides of (4.20) and (4.21), we have the following relationships:

$$0 = -(PN_1 + P\bar{N}_1 + I)\mathbb{E}[Z] - P(C^\top + \bar{C}^\top)\mathbb{E}[k] + \mathbb{E}[\beta], \quad (4.24)$$

$$0 = -(PN_2 + \bar{P}N_2 + I)\mathbb{E}[R - P(D^\top + \bar{D}^\top)\mathbb{E}[k] + \mathbb{E}[\Phi], \quad (4.25)$$

$$0 = -(PN_1(t) + I)(Z - \mathbb{E}[Z]) - PC^\top(k - \mathbb{E}[k]) + (\beta - \mathbb{E}[\beta]), \quad (4.26)$$

$$0 = -(PN_2 + I)(R - E[R]) - PD^\top(k - \mathbb{E}[k]) + \Phi - \mathbb{E}[\Phi]. \quad (4.27)$$

Assume that $PN_1 + P\bar{N}_1 + I, PN_2 + \bar{P}N_2 + I, PN_1 + I$ are invertible, we get

$$\mathbb{E}[Z] = -(PN_1 + P\bar{N}_1 + I)^{-1} \left\{ P(C^\top + \bar{C}^\top)\mathbb{E}[k] - \mathbb{E}[\beta] \right\}, \quad (4.28)$$

$$\mathbb{E}[R] = -(PN_2 + \bar{P}N_2 + I)^{-1} \left\{ P(D^\top + \bar{D}^\top)\mathbb{E}[k] - \mathbb{E}[\Phi] \right\}, \quad (4.29)$$

$$(Z - \mathbb{E}[Z]) = -(PN_1 + I)^{-1} \left\{ PC^\top(k - \mathbb{E}[k]) - (\beta - \mathbb{E}[\beta]) \right\}, \quad (4.30)$$

$$(R - E[R]) = -(PN_2 + I)^{-1} \left\{ PD^\top(k - \mathbb{E}[k]) - (\Phi - \mathbb{E}[\Phi]) \right\}. \quad (4.31)$$

Putting (4.30) and (4.31) into (4.23), we get that

$$\begin{aligned} & \left(\dot{P} - PA^\top - AP - PQP + BN_3^{-1}B^\top + C(PN_1 + I)^{-1}PC^\top + \int_E D(PN_2 + I)^{-1}PD^\top \nu(de) \right) (k - \mathbb{E}[k]) \\ & - C(PN_1 + I)^{-1}(\beta - \mathbb{E}[\beta]) - \int_E D(PN_2 + I)^{-1}(\Phi - \mathbb{E}[\Phi])\nu(de) - (A + PQ)\eta(t) + \gamma = 0. \end{aligned} \quad (4.32)$$

from which one should let

$$\begin{cases} \dot{P} - PA^\top - AP - PQP + BN_3^{-1}B^\top + C(PN_1 + I)^{-1}PC^\top + \int_E D(PN_2 + I)^{-1}PD^\top \nu(de) = 0, \\ \gamma - C(PN_1 + I)^{-1}(\beta - \mathbb{E}[\beta]) - \int_E D(PN_2 + I)^{-1}(\Phi - \mathbb{E}[\Phi])\nu(de) - (A + PQ)\eta = 0. \end{cases} \quad (4.33)$$

Furthermore, from (4.16) and (4.12), we have

$$\begin{aligned} & \left\{ (A + \bar{A})\mathbb{E}[Y] - (B + \bar{B})(N + \bar{N})^{-1}(B + \bar{B})^\top \mathbb{E}[k] + (C + \bar{C})\mathbb{E}[Z] + \int_E (D + \bar{D})\mathbb{E}[R]\nu(de) \right\} ds \\ & = d\mathbb{E}[Y] \\ & = d\Pi\mathbb{E}[k] + d\mathbb{E}[\varphi] \\ & = \left[\dot{\Pi}\mathbb{E}[k] - \Pi(A^\top + \bar{A}^\top)\mathbb{E}[k] - \Pi(Q + \bar{Q})\mathbb{E}[Y] + \mathbb{E}[\alpha] \right] ds. \end{aligned} \quad (4.34)$$

Putting (4.16), (4.28) and (4.29) into the left hand of the above equality and comparing both sides of the above equality, we get

$$\begin{aligned}
0 = & \left\{ \dot{\Pi} - \Pi(A^\top + \bar{A}^\top) - (A + \bar{A})\Pi - \Pi(Q + \bar{Q})\Pi + (B + \bar{B})(N_3 + \bar{N}_3)^{-1}(B + \bar{B})^\top \right. \\
& + (C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1}P(C^\top + \bar{C}^\top) + \int_E (D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1}P(D^\top + \bar{D}^\top) \Big\} k \\
& + \left[-\Pi(Q + \bar{Q}) - (A + \bar{A}) \right] \mathbb{E}[\varphi] + \mathbb{E}[\alpha] - (C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1}\mathbb{E}[\beta] \\
& - \int_E (D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1}\mathbb{E}[\Phi]\nu(de).
\end{aligned} \tag{4.35}$$

Hence, one should let

$$\left\{ \begin{aligned} & \dot{\Pi} - \Pi(A^\top + \bar{A}^\top) - (A + \bar{A})\Pi - \Pi(Q + \bar{Q})\Pi + (B + \bar{B})(N_3 + \bar{N}_3)^{-1}(B + \bar{B})^\top \\ & + (C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1}P(C^\top + \bar{C}^\top) + \int_E (D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1}P(D^\top + \bar{D}^\top) = 0, \\ & \mathbb{E}[\alpha] - [\Pi(Q + \bar{Q}) + (A + \bar{A})]\mathbb{E}[\varphi] - (C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1}\mathbb{E}[\beta] - \int_E (D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1}\mathbb{E}[\Phi]\nu(de). \end{aligned} \right. \tag{4.36}$$

Moreover, comparing the terminal values on both sides of the two equations in (4.16) and (4.17), one has

$$\Pi(T) = 0, P(T) = 0.$$

Therefore, by (4.33) and (4.36), we see that $P(\cdot)$ and $\Pi(\cdot)$ should satisfy the following Riccati-type equations, respectively:

$$\left\{ \begin{aligned} & \dot{P} - PA^\top - AP - PQP + BN_3^{-1}B^\top + C(PN_1 + I)^{-1}PC^\top + \int_E D(PN_2 + I)^{-1}PD^\top \nu(de) = 0, \\ & P(T) = 0, \end{aligned} \right. \tag{4.37}$$

and

$$\left\{ \begin{aligned} & \dot{\Pi} - \Pi(A^\top + \bar{A}^\top) - (A + \bar{A})\Pi - \Pi(Q + \bar{Q})\Pi + (B + \bar{B})(N_3 + \bar{N}_3)^{-1}(B + \bar{B})^\top \\ & + (C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1}P(C^\top + \bar{C}^\top) + \int_E (D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1}P(D^\top + \bar{D}^\top) = 0, \\ & \Pi(T) = 0, \end{aligned} \right. \tag{4.38}$$

and $\varphi(t)$ should satisfy the following MF-BSDE on $[0, T]$:

$$\left\{ \begin{aligned} d\varphi = & \left\{ (PQ + A)\varphi + C(PN_1 + I)^{-1}\beta + \int_E D(PN_2 + I)^{-1}\Phi\nu(de) \right. \\ & + (\bar{A} + \Pi(Q + \bar{Q}) - PQ)\mathbb{E}[\varphi] + [(C + \bar{C})(PN_1 + P\bar{N}_1 + I)^{-1} - C(PN_1 + I)^{-1}]\mathbb{E}[\beta] \\ & + \int_E [(D + \bar{D})(PN_2 + P\bar{N}_2 + I)^{-1} - D(PN_2 + I)^{-1}]\mathbb{E}[\Phi]\nu(de) \Big\} ds \\ & + \beta dW + \Phi d\tilde{\mu}(de, ds), \\ \varphi(T) = & \xi. \end{aligned} \right. \tag{4.39}$$

By the same argument as in section 4 of [11], under Assumptions 2.1 and 2.2, we can get that Riccati equations (4.37) and (4.38) have a unique solution, respectively.

4.3 Representations of optimal controls

This section is going to give explicit formulas of the optimal controls and the value function, via the solutions to the Riccati equations (4.37), (4.38), and the MF-BSDE (4.39). Now we state our main result as follows.

Theorem 4.3. *Let Assumption 2.1 and 2.2 be satisfied. Let $P(\cdot)$ and $\Pi(\cdot)$ be the unique solutions to the Riccati equations (4.37) and (4.38), respectively, and let $(\varphi(\cdot), \beta(\cdot), \Phi(\cdot))$ be the unique adapted solution to the MF-BSDE (4.39). Then the following MF-FSDE admits a unique solution $k(\cdot)$:*

$$\left\{ \begin{array}{l} dk = - \left[(A^\top + QP)k + (\bar{A}^\top - QP + (Q + \bar{Q})\Pi)\mathbb{E}[k] + Q\varphi + \bar{Q}\mathbb{E}[\varphi] \right] ds - \left[[I - N_1(PN_1 + I)^{-1}P]C^\top k \right. \\ \quad + [\bar{C}^\top + N_1(PN_1 + I)^{-1}PC^\top - (N_1 + \bar{N}_1)(PN_1 + P\bar{N}_1 + I)^{-1}P(C^\top + \bar{C}^\top)]\mathbb{E}[k] \\ \quad \left. - N_1(PN_1 + I)^{-1}(\beta - \mathbb{E}[\beta]) - (N_1 + \bar{N}_1)(PN_1 + P\bar{N}_1 + I)^{-1}\mathbb{E}[\beta] \right] dW \\ \quad - \int_E \left[[I - N_2(PN_2 + I)^{-1}P]D^\top k + [\bar{D}^\top + 2N_2(PN_2 + I)^{-1}PD^\top \right. \\ \quad \left. - (\bar{N}_2 + \bar{N}_2)(PN_2 + P\bar{N}_2 + I)^{-1}P(D^\top + \bar{D}^\top)]\mathbb{E}[k] \right. \\ \quad \left. - N_2(PN_2 + I)^{-1}(\Phi - \mathbb{E}[\Phi]) - (N_2 + \bar{N}_2)(PN_2 + P\bar{N}_2 + I)^{-1}\mathbb{E}[\Phi] \right] \tilde{\mu}(d\theta, ds), s \in [t, T], \\ k(t) = -(I + GP)^{-1}G(\varphi(t) - \mathbb{E}[\varphi(t)]) - (I + (G + \bar{G})\Pi)^{-1}(G + \bar{G})\mathbb{E}[\varphi(t)], \end{array} \right. \quad (4.40)$$

and the unique optimal control of Problem 2.1 for the terminal state ξ is given by

$$u = -N_3^{-1}B^\top [k + \mathbb{E}[k]] - N_3^{-1}\bar{N}_3(N_3 + \bar{N}_3)^{-1}(B + \bar{B})^\top \mathbb{E}[k], \quad a.e.a.s. \quad (4.41)$$

Proof. Let $P(\cdot)$ and $\Pi(\cdot)$ be the unique solutions to the Riccati equations (4.37) and (4.38), respectively, and let $(\varphi(\cdot), \beta(\cdot), \Phi(\cdot))$ be the unique adapted solution to the MF-BSDE (4.39). It is clear that (4.40) has a unique solution $k(\cdot)$. So we need only prove that $u(\cdot)$ defined by (4.41) is the unique optimal control of Problem 2.1 for the terminal state ξ . To this end, define

$$Y =: P(k - \mathbb{E}[k]) + \Pi\mathbb{E}[k] + \varphi, \quad (4.42)$$

$$Z =: -(PN_1 + I)^{-1} \left[PC^\top (k - \mathbb{E}[k]) - (\beta - \mathbb{E}[\beta]) \right] - (PN_1 + P\bar{N}_1 + I)^{-1} \left[P(C^\top + \bar{C}^\top)\mathbb{E}[k] - \mathbb{E}[\beta] \right], \quad (4.43)$$

$$R =: -(PN_2 + I)^{-1} \left[PD^\top (k - \mathbb{E}[k]) - (\Phi - \mathbb{E}[\Phi]) \right] - (PN_2 + P\bar{N}_2 + I)^{-1} \left[P(D^\top + \bar{D}^\top)\mathbb{E}[k] - \mathbb{E}[\Phi] \right]. \quad (4.44)$$

Then following the derivation of riccati equation in the previous subsection, by applying Itô formula to $Y(s) = P(s)(k(s) - \mathbb{E}[k(s)]) + \Pi(s)\mathbb{E}[k(s)] + \varphi(s)$, it is easy to check that $(u(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot), k(\cdot))$ satisfies the following stochastic Hamilton system

$$\left\{ \begin{array}{l} dY(s) = \left\{ A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)] \right. \\ \quad \left. - \int_E D(s, e)R(s, e)\nu(de) + \int_E \bar{D}(s, e)\mathbb{E}[R(s, e)]\nu(de) \right\} ds + Z(s)dW(s) + \int_E R(t, e)\tilde{\mu}(d\theta, ds), \\ dk(s) = - \left[A(s)^\top k(s) + \bar{A}(s)^\top \mathbb{E}[k(s)] + Q(s)Y(s) + \bar{Q}(s)\mathbb{E}[Y(s)] \right] ds \\ \quad - \left[C(s)^\top k(s) + \bar{C}(s)^\top \mathbb{E}[k(s)] + N_1(s)Z(s) + \bar{N}_1(s)\mathbb{E}[Z(s)] \right] dW(s) \\ \quad - \int_E \left[D(s, e)^\top k(s) + \bar{D}(s, e)^\top \mathbb{E}[k(s)] + N_2(s, e)R(s, e) + \bar{N}_2(s, e)\mathbb{E}[R(s, e)] \right] \tilde{\mu}(d\theta, ds), \\ Y(T) = \xi, k(t) = -GY(t) - \bar{G}\mathbb{E}[Y(t)], \\ N_3(s)u(s) + \bar{N}_3(s)\mathbb{E}[u(s)] + B^\top(s)k(s) + \bar{B}^\top(s)\mathbb{E}[k(s)] = 0, \quad s \in [t, T]. \end{array} \right. \quad (4.45)$$

Thus, by Theorem 4.2, we know that $u(\cdot)$ defined by (4.41) is the unique optimal control of Problem 2.1. The proof is complete. \square

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